SPECTRAL SPACES AND KRULL DIMENSION

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Spectral spaces are sober topological spaces with the following property: the quasi-compact open subsets are stable under finite intersection and form a basis for the topology. Hochster [7] has shown that any spectral space arises as the spectrum of a commutative ring. It is thus tempting to infer classical concepts of commutative ring theory into the category of spectral spaces, and to study to which extent ring-theoretical properties “survive” in this topological context. The aim of this memoir is to investigate the concept of Krull dimension under this perspective.

Minimal and maximal prime ideals of a ring correspond to generic and closed points of its spectrum. Inclusion of prime ideals induces the so-called specialisation order on the points of the spectrum. Accordingly, the Krull dimension of a spectral space is defined to be the length of longest specialisation chain of points.

Spectral spaces usually do not satisfy Hausdorff’s separation axiom. Those spectral spaces which do are very special: they are called Boolean spaces and can also be characterised as those spectral spaces which have Krull dimension 0, or as those which correspond to Boolean lattices under Stone duality, cf. [1].

Every commutative ring possesses a universal extension of Krull dimension 0 (these are also called absolutely flat or von Neumann regular). Dually, every spectral space possesses a universal Booleanisation, obtained by replacing the spectral topology with the patch (aka constructible) topology. This leads to a natural notion of relative Krull dimension of a map of spectral spaces, by dualising the definition of Lombardi and Quitté for commutative ring homomorphisms, cf. [9, Sct. XIII.7].

There are research articles concerned with the Krull dimension of spectral spaces. The interested student may choose one among the following three directions:

In [3] the authors show that for a normal spectral space, the Krull dimension is an upper bound for the covering dimension. A spectral space is normal if and only if it is the spectrum of a Gelfand ring, cf. [10]. In [4] the authors show that for any spectral space, Krull dimension and small inductive dimension coincide.

In [2] the authors define a Hausdorff dimension for any constructible subspace $Y$ of a spectral space $X$, and show that it is a finite invariant measuring the “complexity” of $Y$ inside the Booleanisation of $X$. This Hausdorff dimension is closely related to the relative Krull dimension of the inclusion $Y \hookrightarrow X$, cf. [8, 9].

In [5], the author introduces and studies “scaled lattices”, which are expansions of distributive lattices by a finite number of unary operations encoding information about the Krull dimension of the dual spectral space. This work has applications both to the model theory of Heyting algebras (which model intuitionistic logic) and of algebraic geometry over countable p-adically closed fields. See also [6] for a published version including these model theoretic applications.

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